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A new algorithm for moving boundary problems subject to periodic boundary conditions

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Abstract

Purpose – The paper aims to introduce a new algorithm based on the boundary integral method developed to solve moving boundary phase change problem subject to all possible cases of cyclic boundary temperature.

Design/methodology/approach – In the present paper, the phase change problem with periodic boundary temperature, which may be above or below the phase change temperature, is analyzed. The analysis is based on applying the boundary integral method in a new numerical algorithm. There are two main topics of the analysis herein. The first one is to study the direct effect of the cyclic boundary temperature on the movement of the moving boundary for various Stefan numbers. The second one is check that the proposed method covers all possible cases of cyclic boundary temperature with respect to the phase change temperature.

Findings – When using the proposed method, it is found an easy mathematical manipulation and the results can be improved when fine time step size used.

Originality/value – The proposed method is a very new method, which can be applied to any case of moving boundary phase change problem subject to any case of cyclic boundary temperature. Also the proposed method takes into consideration different parameters that affect directly on the evolution of the moving boundary such as Stefan number, etc.

Keywords Boundary layers, Temperature, Phase transformations

Paper type Research paper

1. Introduction

Phase change problems occur in many areas of practical interest such as the metal, glass, plastic and oil industries, space vehicles design, preservation the food and plasma-physics. Phase change problems with time-dependent boundary conditions require special numerical techniques (Furzeland, 1980; Gupta and Kumar, 1980, 1981; Menning and Özisik, 1985; Yao and Prusa, 1989; Samarskii *et al.*, 1993), requiring small time steps and small spatial grid size for accurate calculation (Fox, 1975).

Phase change problem subjected to periodic boundary conditions are important for practical cases involving melting and solidification of ice and industrial processes with cyclical surface temperature or heat flux variation. For such applications, both the moving boundary and the internal temperature distribution are very important. Rizwan (1999) developed a nodal integral method (NIM), in which, both time and spatial domain were discretized, then using a special transformation to fix the domain and finally, integrate in both directions simultaneously.

In the present paper, the phase change problem with periodic boundary temperature, which may be above or below the phase change temperature, is analyzed. The analysis is

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based on applying the boundary integral method (Ahmed and Wrobel, 1995; Ahmed, 1997; Ahmed and Megahed, 1998) in a new numerical algorithm.

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2. Problem description and formulation

Consider one-dimensional phase change problem bounded by a fixed boundary $x = 0$ and the other side is free to move as a function of time, $x = s(t)$. The problem can be recast as follows:

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} \quad 0 \leq x \leq s(t) \tag{1}$$

$$T(x,t) = f(t) \quad x = 0 \tag{2}$$

$$T(x,t) = T_{\mathrm{pc}} \quad x = s(t) \tag{3}$$

$$T(x,t) = T_i \quad t = 0 \tag{4}$$

$$\frac{\partial T(x,t)}{\partial x} = -Ste \frac{\mathrm{d}x(t)}{\mathrm{d}t} \quad x = s(t) \tag{5}$$

In the above set of equations, $f(t)$ represents the Dirichlet periodic input temperature at the boundary $x = 0$, and Ste is the Stefan number.

3. Boundary integral method

3.1 Case (1): fixed domain formulation

For fixed domain problem bounded by $x = 0$ and $x = \ell$, where ℓ is a truncated long enough boundary, with Dirichlet boundary condition at $x = 0$ and Neumann boundary condition at $x = \ell$, the boundary integral formulation for equation (1) can be recast in the following general form as follows (Ahmed and Wrobel, 1995):

$$C(\zeta)T(\zeta, t_F) = \int_{t_o}^{t_F} \left\{ T^*(\zeta, x, t, t_F) \frac{\partial T(x,t)}{\partial x} - T(x,t) \frac{\partial T^*(\zeta, x, t, t_F)}{\partial x} \right\}_0^{\ell} \mathrm{d}t + \int_0^{\ell} T(x,0) T^*(\zeta, x, 0, t_F)\, \mathrm{d}x \tag{6}$$

3.2 Case (2): moving domain formulation

In this formulation the set of equations (1)-(5) are used in the boundary integral formulation. According to Shaw (1982) and Wrobel (1983) this set of equations can be recast in the following boundary integral equation in its general form:

$$C(\zeta)T(\zeta,t_F)=\int_{t_o}^{t_F}\left\{T^*(\zeta,x,t,t_F)\frac{\partial T(x,t)}{\partial x}-T(x,t)\frac{\partial T^*(\zeta,x,t,t_F)}{\partial x}\right. \left.+T(x,t)T^*(\zeta,x,t,t_F)\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right\}_0^{s(t)}\mathrm{d}t \tag{7}$$

Equations (6) and (7) are called the boundary integral formulas for fixed and moving problem, respectively. In these equations the point ζ is called the source point and the coefficient $C(\zeta)$ can only take the value of 0.5 for a boundary point and 1 for an internal point.

4. Development the proposed algorithm

The proposed algorithm is based on treating the location of the moving boundary once as a boundary point, when applying the moving domain boundary integral code, and once more as an internal point when applying the fixed domain boundary integral code. The systematic diagram for the proposed procedure at two successive time steps is shown in Figure 1.

4.1 The main steps of the proposed method can be summarized as follow

- At the first time step t_j, guess an initial position of the moving boundary $s^{i=1}(t_j)$ and apply the moving boundary integral code (MBIC) and iterate until the Stefan condition satisfied at a position $s^{n_1}(t_j)$ where n_1 is the total number of iteration t_j.
- At the second time step t_{j+1} treat the position $s^{n_1}(t_j)$ as an internal point and use the fixed boundary integral code (FBIC) to find the temperature at that position. Then guess another position $s^i(t_{j+1})$ and use (MBIC) to obtain $s^{n_2}(t_{j+1})$ where n_2 is the total number of iteration at time t_{j+1}.
- Repeat the second step for each time step and stop at the final time step t_F, see Figure 2, taking into consideration that when the time becomes between $\omega/2$ and ω, the second moving boundary starts appearing.

5. Numerical results and discussion

A one-dimensional phase change problem subjected to a Dirichlet periodic boundary condition temperature is analyzed. The time-dependent surface temperature introduces the surface temperature oscillation amplitude ε and frequency ω as two additional parameters.

The present algorithm had been tested for different values of Stefan's number because the growth of the moving boundary depends very strongly upon it. Both the amplitude and the frequency of the surface boundary temperature are kept constant throughout the first case study of the analysis. The numerical values used for the first case study taken from Rizwan (1999), are shown in Table I.

In the second case study (Voller *et al.*, 1996), the cyclic boundary temperature is of the form $T=T_m+\varepsilon\sin(\omega t)$. In this formula, the melting temperature $T_m=0$, $\omega=\pi/10$ and four different cases for the amplitude, 50, 100, 150 and 200 are tested. All of these amplitudes will result a boundary temperature that will oscillate above and below the phase change temperature.

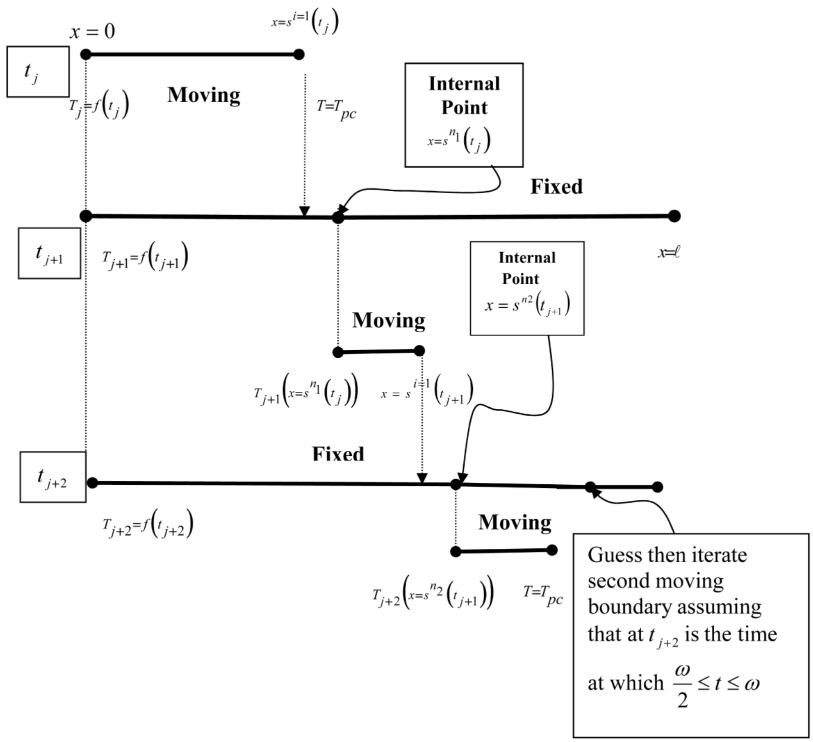

Figure 1.
Systematic diagram for the proposed method

5.1 First case study

The problem under consideration was solved using the NIM and its results can be found in Rizwan (1999). A comparison between the nodal and boundary integral methods is made. The comparison concentrated first on the evolution of the moving boundary at different Stefan number as shown in Figures 3-5.

It is clear that the oscillating boundary temperature impacts the growth rate of the moving boundary and these impact decreases by increasing the time and the moving boundary evolves as the square root of the time (Rizwan, 1999). Stefan's number affects strongly on the evolution of the moving boundary, i.e. by increasing the this number and at the same time step the growth rate of the moving boundary goes very quickly.

It is also clear from these figures that there is some difference between the NIM and the boundary integral method but this difference decreases by increasing the time. This difference comes from the approximate nature of the two methods in addition to the iterations procedure within each time step occurred in the present method, but it can be reduced if a small time step is used and the prescribed error decreased.

The temperature distribution based on the present method is evaluated at a time $t = 16$ and at different three values of Stefan number as shown in Figure 6.

5.2 Second case study

In this case and for frequency $\omega = \pi/10$, the time period divided into two parts, the first part $0 \leq t < 10$ the material melt, then freeze for $10 \leq t < 20$. In the first interval

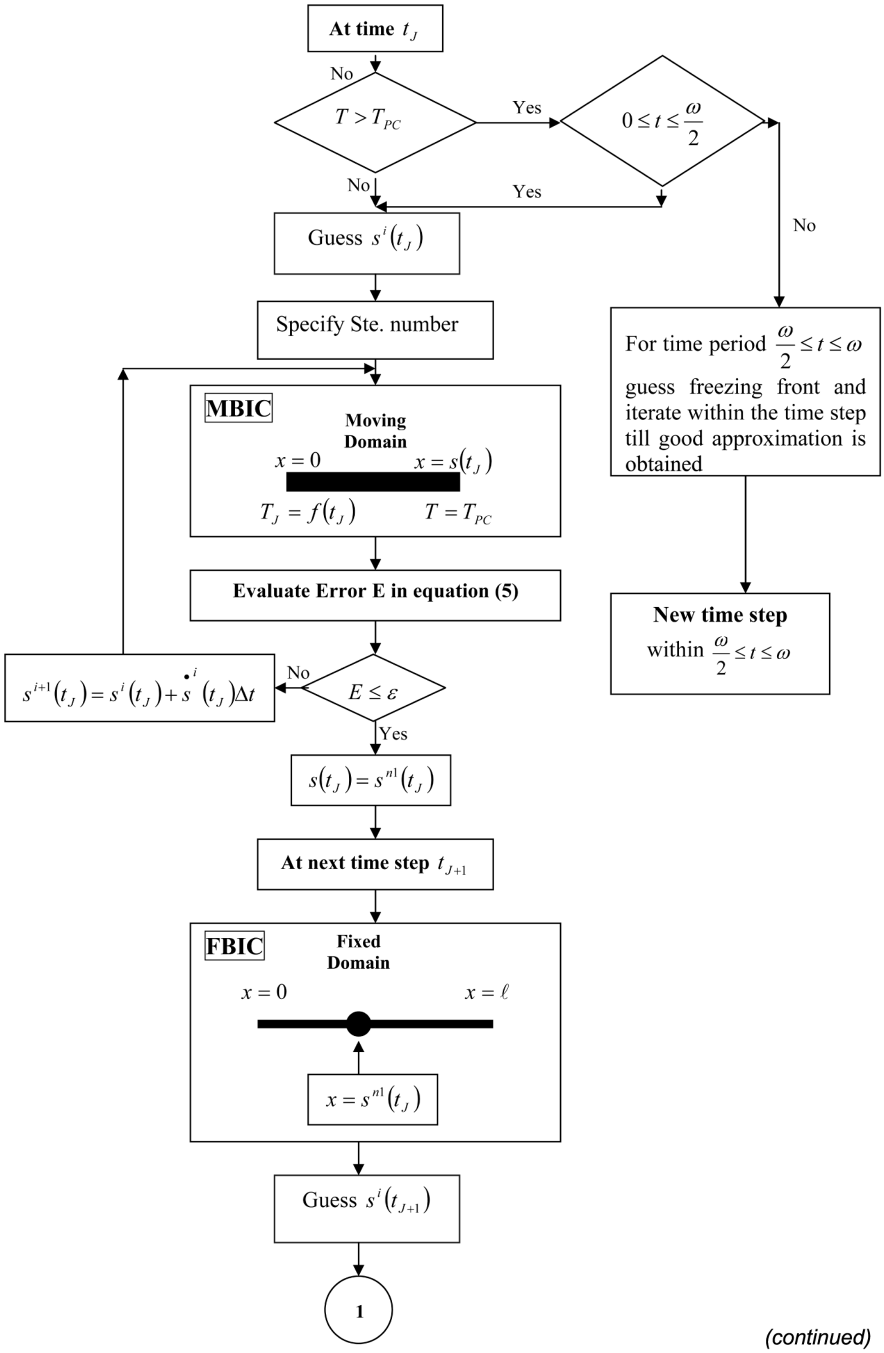

Figure 2.
Flow chart

(continued)

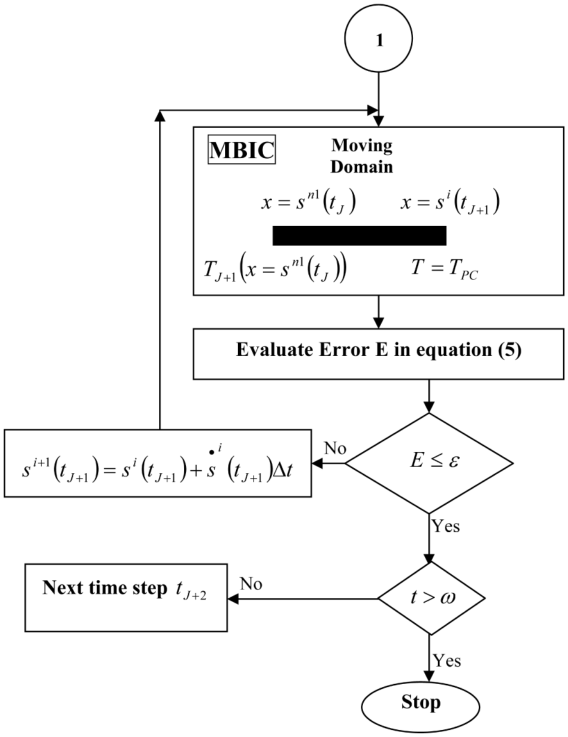

Figure 2.

Table I. Numerical values

Input surface	ε	ω	Stefan's number		
$T = 1 + \varepsilon \sin \omega t$	0.5	$\pi/2$	0.02	1	2

the melting front only appears, while in the second interval both melting and freezing coexist. This case of study were solved for different amplitude ranging from 50 to 200 and the results are shown Figures 7 and 8.

6. Conclusion

In the present paper, the one-dimensional phase change problem with periodically oscillating temperature on the fixed boundary is analyzed. The analysis is based on the boundary integral formulation that incorporates a new numerical algorithm for handling the oscillating boundary condition. In a case where the melting/solidification was monotonic the results using this algorithm clearly show the effects of the oscillating boundary temperature on the early development of the phase change and the effect of the Stefan number on controlling the phase change rate. The algorithm was also applied to a problem where the boundary temperature oscillated both above and below the phase change temperature. Results on this problem predicted simultaneous melting and solidification fronts in the domain. For both test problems

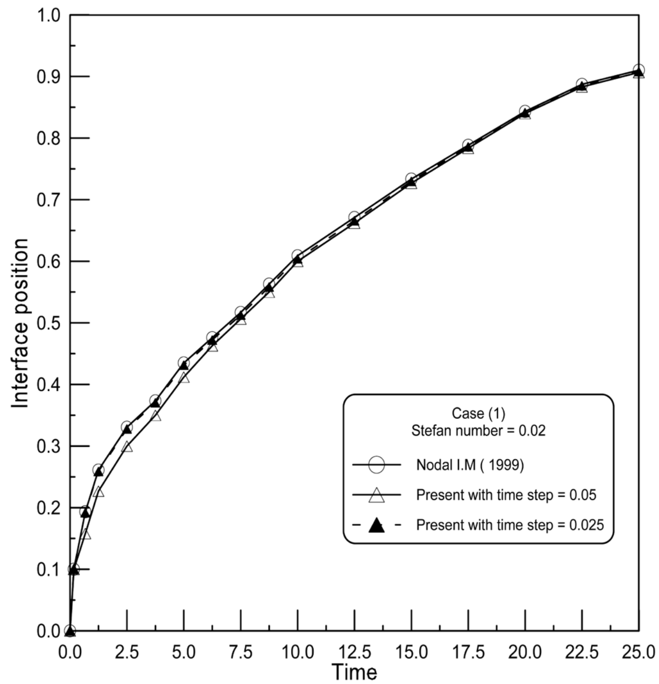

Figure 3.
Moving boundary at
Stefan number = 0.02

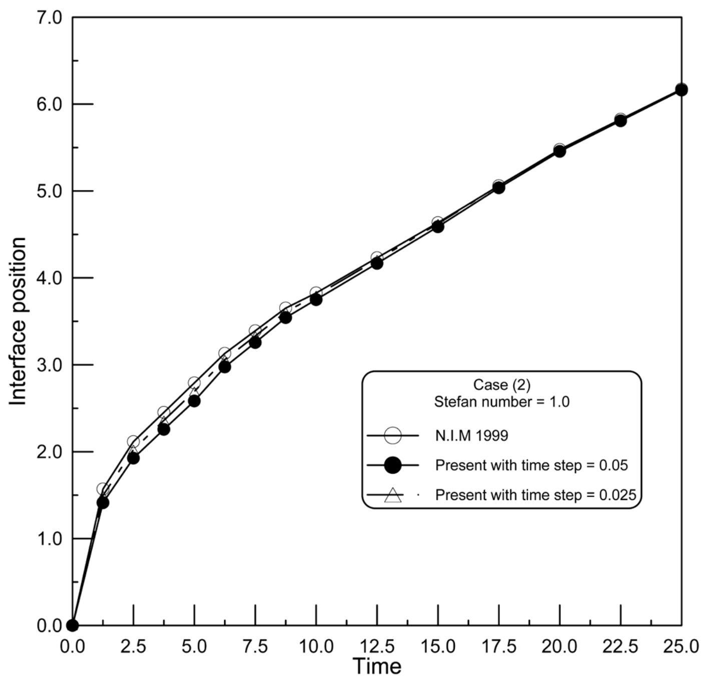

Figure 4.
Moving boundary at
Stefan number = 1.0

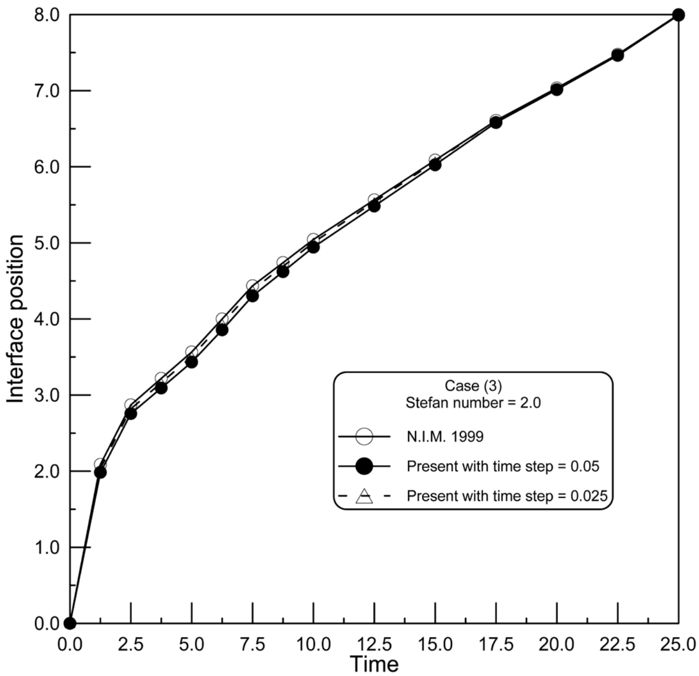

Figure 5.
Moving boundary at Stefan number = 2.0

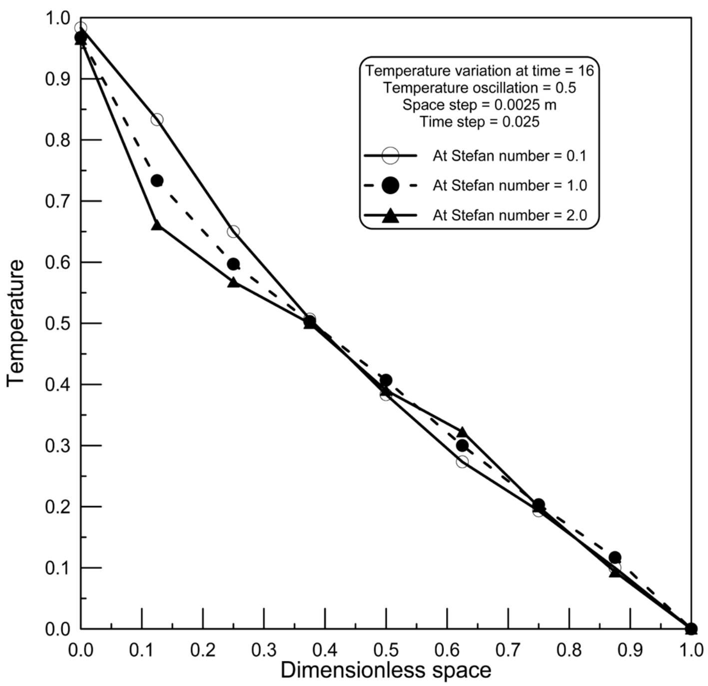

Figure 6.
Temperature variation at different Stefan number

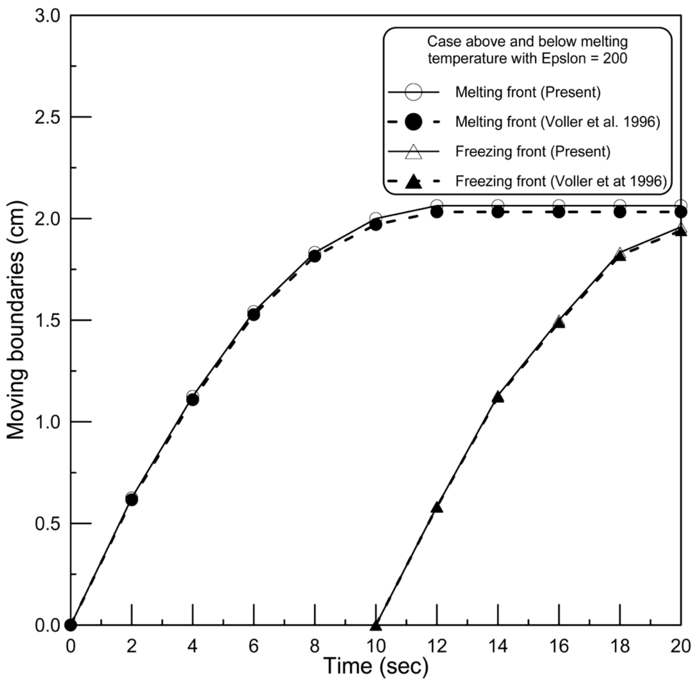

Figure 7.
Case above and below phase change problem with $\varepsilon = 200$

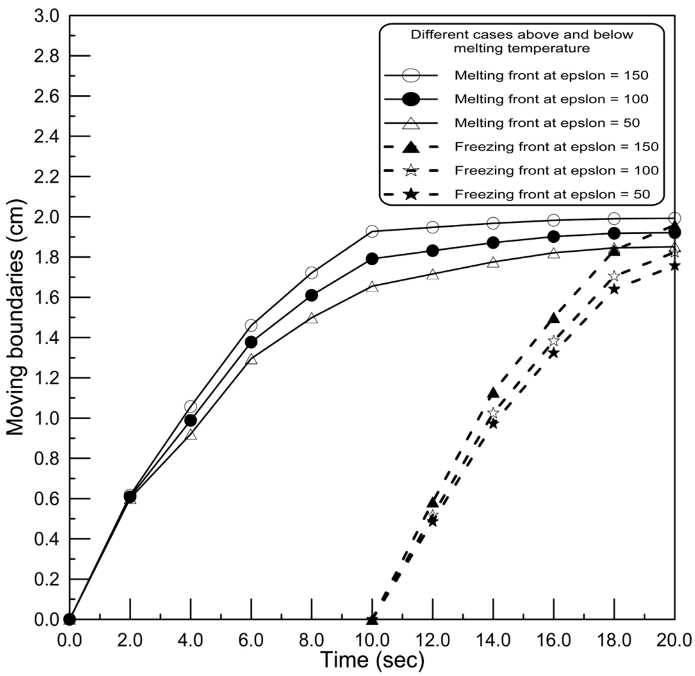

Figure 8.
Case above and below phase change problem at different ε

the results obtained were also in close agreement with prior alternative numerical approaches based on a NIM (Rizwan, 1999) and an enthalpy method (Voller *et al.*, 1996).

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